CONVECTIVE DIFFUSION IN AN INFINITE BAND

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A solution of the mixed boundary-value problem in the form of a generalized function represented by a contour integral is obtained for the stationary equation of convective diffusion.

We simulate the cathode region of a chemotron converter by an infinite band $P\{|\alpha| \leq 1, |\beta| < \infty\}$, where α and β are real dimensionless variables. We assume that an electrolyte flows along the band at a constant velocity v from $+\infty$ to $-\infty$ with the velocity vector parallel to the axis of the band. The boundary of the half-band $|\alpha| \leq 1$, $\beta > 0$, i.e., the rays $\alpha = \pm 1$, $\beta > 0$, is assumed to be made of an insulating, chemically inert material and the rays $\alpha = \pm 1$, $\beta < 0$ act as a cathode which is indifferent with respect to the solution used in the instrument.

For these conditions, we shall formulate the problem of determining the concentration distribution $C(\alpha, \beta)$ of the reducible component of the electrolyte in the cathode region described above for a steady-state process. In this case, calculation of reagent concentration reduces to integration of the equation for stationary convective diffusion, which can be written in dimensionless form in the following manner:

$$-\gamma \frac{\partial U}{\partial \beta} = \frac{\partial^2 U}{\partial \alpha^2} + \frac{\partial^2 U}{\partial \beta^2}, \tag{1}$$

where $U = C(\alpha, \beta)/C_0$; C_0 is the concentration of the reducible component of the electrolyte at the entrance to the cathode channel, i.e., $C_0 = \lim_{\beta \to +\infty} C(\alpha, \beta)$. We also make the reasonable assumption [2] that the reagent concentration is zero at the exit from the cathode channel, i.e., that $\lim_{\beta \to -\infty} C(\alpha, \beta) = 0$. These assumptions make it possible to establish the conditions which the function $U(\alpha, \beta)$ must satisfy at infinity:

$$\lim_{\beta \to +\infty} U(\alpha, \beta) = 1, \text{ and } \lim_{\beta \to -\infty} U(\alpha, \beta) = 0.$$

One can now formulate the problem more precisely. It is necessary to solve Eq. (1) under the following boundary conditions:

$$\left.\frac{\partial U}{\partial \alpha}\right|_{\alpha = \pm 1} = 0, \quad \beta > 0, \tag{2}$$

$$U(\pm 1, \beta) = 0, \quad \beta < 0, \tag{3}$$

$$\lim_{\beta \to +\infty} U(\alpha, \beta) = 1, \quad |\alpha| < 1, \tag{4}$$

$$\lim_{\beta \to -\infty} U(\alpha, \beta) = 0, \quad |\alpha| < 1. \tag{5}$$

We shall seek a solution of this problem in the form of the difference

$$U(\alpha, \beta) = 1 - S(\alpha, \beta),$$

where $S(\alpha, \beta)$ is a solution of Eq. (1) satisfying the boundary conditions

$$\left.\frac{\partial S}{\partial \alpha}\right|_{\alpha = \pm 1} = 0, \quad \beta > 0, \tag{6}$$

$$S(\pm 1, \beta) = 1, \quad \beta < 0, \tag{7}$$

$$\lim_{\beta \to +\infty} S(\alpha, \beta) = 0, \quad |\alpha| < 1, \tag{8}$$

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$$\lim_{\beta \to -\infty} S(\alpha, \beta) = 1, \quad |\alpha| < 1. \tag{9}$$

It is easy to verify that the function

$$Q(\alpha, \beta) = \int_{-i\infty}^{+i\infty} A(\omega) e^{\left(\omega - \frac{\gamma}{2}\right)\beta} \cos \sqrt{\omega^2 - \frac{\gamma^2}{4}}\, \alpha d\omega \tag{10}$$

is an even solution of Eq. (1) with respect to the variable α. In Eq. (10), $A(\omega)$ is an as yet arbitrary function of the complex variable ω, which is analytic on the imaginary axis of the ω plane, and for the root $\sqrt{\omega^2 - \gamma^2/4}$ that branch is chosen which, for real ω such that $\omega^2 - \gamma^2/4 \geq 0$, gives the arithmetic value of the root.

We require that the function $Q(\alpha, \beta)$ satisfy conditions (6) and (7). Accordingly, this leads to the two integral equations

$$\int_{-i\infty}^{+i\infty} A(\omega) \sqrt{\omega^2 - \frac{\gamma^2}{4}} \sin \sqrt{\omega^2 - \frac{\gamma^2}{4}}\, e^{\left(\omega - \frac{\gamma}{2}\right)\beta} d\omega = 0, \qquad \beta > 0, \tag{11}$$

$$\int_{-i\infty}^{+i\infty} A(\omega) \cos \sqrt{\omega^2 - \frac{\gamma^2}{4}}\, e^{\left(\omega - \frac{\gamma}{2}\right)\beta} d\omega = 1, \qquad \beta < 0. \tag{12}$$

We introduce the notation

$$K(\omega) = A(\omega) \sqrt{\omega^2 - \frac{\gamma^2}{4}} \sin \sqrt{\omega^2 - \frac{\gamma^2}{4}},$$

$$\psi(\omega) = \frac{K(\omega) \cos \sqrt{\omega^2 - \frac{\gamma^2}{4}}}{\sqrt{\omega^2 - \frac{\gamma^2}{4}} \sin \sqrt{\omega^2 - \frac{\gamma^2}{4}}}.$$

Equations (11) and (12) then take the form

$$\int_{-i\infty}^{+i\infty} K(\omega) e^{\left(\omega - \frac{\gamma}{2}\right)\beta} d\omega = 0, \quad \beta > 0, \tag{13}$$

$$\int_{-i\infty}^{+i\infty} \psi(\omega) e^{\left(\omega - \frac{\gamma}{2}\right)\beta} d\omega = 1, \quad \beta < 0. \tag{14}$$

We introduce the function

$$\Pi(\omega) = \prod_{n=1}^{\infty} \frac{\left(1 - \frac{\omega}{a_n}\right)}{\left(1 - \frac{\omega}{b_n}\right)}, \tag{15}$$

where the a_n are those positive roots of the function $\sin\sqrt{\omega^2 - \gamma^2/4}$ which are greater than the root $\omega = \gamma/2$, and the b_n are the positive roots of the function $\cos\sqrt{\omega^2 - \gamma^2/4}$.

Following Danilevskii [1], one can show that the product (15) converges uniformly and absolutely everywhere in the region $|\arg\omega| \geq \delta$, where δ is a fixed positive number as small as desired ($\delta < \pi/2$).

Investigating the behavior of $\Pi(\omega)$ at infinity, we find that

$$\Pi(\omega) \sim \frac{1}{i\sqrt{\omega}}. \tag{16}$$

From the method by which the meromorphic function $\Pi(\omega)$ was constructed and from the asymptotic behavior (16), it follows that this function is regular in the region $|\arg\omega| \geq \delta$ and goes to zero uniformly in this region when $\omega \to \infty$.

We take the function $B\Pi(\omega)$, where $B = \text{const}$, as $K(\omega)$. Such a choice is possible since the arbitrary function $A(\omega)$ appears in $K(\omega)$. Equation (13) can now be written as

$$\int_{-i\infty}^{+i\infty} B\Pi(\omega) e^{\left(\omega-\frac{\gamma}{2}\right)\beta} d\omega = 0, \quad \beta > 0. \tag{17}$$

Since $\Pi(\omega)$ is regular in the region $\mathrm{Re}\,\omega \leq 0$ and obviously meets the conditions of the Jordan lemma in this region, the condition (13) is satisfied by such a choice of $K(\omega)$.

From the properties of $\Pi(\omega)$ noted, it follows that in the region $\mathrm{Re}\,\omega \geq 0$ the function

$$\psi(\omega) = \frac{B\Pi(\omega)\cos\sqrt{\omega^2-\frac{\gamma^2}{4}}}{\sqrt{\omega^2-\frac{\gamma^2}{4}}\sin\sqrt{\omega^2-\frac{\gamma^2}{4}}}$$

has a singularity at $\omega = \gamma/2$ which is a simple pole. It is easy to see that

$$\mathrm{res}\left[\psi(\omega) e^{-\frac{\gamma\beta}{2}}, \frac{\gamma}{2}\right] = \frac{B\Pi\left(\frac{\gamma}{2}\right) e^{-\frac{\gamma\beta}{2}}}{\gamma}$$

But then

$$\mathrm{res}\left[\psi(\omega) e^{\left(\omega-\frac{\gamma}{2}\right)\beta}, \frac{\gamma}{2}\right] = \frac{B\Pi\left(\frac{\gamma}{2}\right)}{\gamma}.$$

Consequently, if one sets

$$B = -\frac{1}{2\pi i}\cdot\frac{\gamma}{\Pi\left(\frac{\gamma}{2}\right)},$$

then for $\beta < 0$, because $\psi(\omega)$ satisfies the conditions for the Jordan lemma in the region $\mathrm{Re}\,\omega \geq 0$,

$$\int_{-i\infty}^{+i\infty} \psi(\omega) e^{\left(\omega-\frac{\gamma}{2}\right)\beta} d\omega = 1,$$

i.e., the condition (14) is also satisfied.

Thus the function $Q(\alpha, \beta)$ defined by the expression

$$Q(\alpha, \beta) = -\frac{\gamma}{\Pi\left(\frac{\gamma}{2}\right)}\cdot\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Pi(\omega)\cos\sqrt{\omega^2-\frac{\gamma^2}{4}}\,\alpha}{\sqrt{\omega^2-\frac{\gamma^2}{4}}\sin\sqrt{\omega^2-\frac{\gamma^2}{4}}} e^{\left(\omega-\frac{\gamma}{2}\right)\beta} d\omega, \tag{18}$$

is a solution of Eq. (1) which satisfies conditions (6) and (7).

We find $\lim\limits_{\beta\to\pm\infty} Q(\alpha, \beta)$ for $|\alpha| < 1$. Equation (18) can be written in the following manner:

$$Q(\alpha, \beta) = -\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\left[\frac{1}{\omega-\frac{\gamma}{2}}\varphi(\omega, \alpha) - \frac{1}{\omega-\frac{\gamma}{2}}\right] e^{\left(\omega-\frac{\gamma}{2}\right)\beta} d\omega - \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{e^{\left(\omega-\frac{\gamma}{2}\right)\beta}}{\omega-\frac{\gamma}{2}} d\omega, \tag{19}$$

where

$$\varphi(\omega, \alpha) = \frac{\gamma\Pi(\omega)}{\Pi\left(\frac{\gamma}{2}\right)}\cdot\frac{\sqrt{\omega^2-\frac{\gamma^2}{4}}\cos\sqrt{\omega^2-\frac{\gamma^2}{4}}\,\alpha}{\left(\omega+\frac{\gamma}{2}\right)\sin\sqrt{\omega^2-\frac{\gamma^2}{4}}}.$$

The function inside the integral sign in the first integral on the right side of Eq. (19) is obviously a regular function everywhere in the right half-plane $\mathrm{Re}\,\omega \geq 0$, and the integral itself converges uniformly with respect to β on the line $\mathrm{Re}\,\omega = 0$. In such a case, one can assert on the basis of the generalized Lebesgue–Riemann lemma that

$$\lim_{\beta\to\pm\infty}\int_{-i\infty}^{+i\infty}\left[\frac{1}{\omega-\frac{\gamma}{2}}\varphi(\omega, \alpha) - \frac{1}{\omega-\frac{\gamma}{2}}\right] e^{\left(\omega-\frac{\gamma}{2}\right)\beta} d\omega = 0.$$

Thus

$$\lim_{\beta\to\pm\infty} Q(\alpha, \beta) = -\frac{1}{2\pi i}\lim_{\beta\to\pm\infty}\int_{-i\infty}^{+i\infty}\frac{e^{\left(\omega-\frac{\gamma}{2}\right)\beta}}{\omega-\frac{\gamma}{2}}\,d\omega.$$

But

$$\lim_{\beta\to+\infty}\int_{-i\infty}^{+i\infty}\frac{e^{\left(\omega-\frac{\gamma}{2}\right)\beta}}{\omega-\frac{\gamma}{2}}\,d\omega = 0,$$

$$\lim_{\beta\to-\infty}\int_{-i\infty}^{+i\infty}\frac{e^{\left(\omega-\frac{\gamma}{2}\right)\beta}}{\omega-\frac{\gamma}{2}}\,d\omega = -2\pi i.$$

Consequently,

$$\lim_{\beta\to+\infty} Q(\alpha, \beta) = 0, \quad \text{a} \quad \lim_{\beta\to-\infty} Q(\alpha, \beta) = 1.$$

Thus the function $Q(\alpha, \beta)$ defined by Eq. (18) satisfies Eq. (1) and conditions (6)-(9), i.e.,

$$Q(\alpha, \beta) \equiv S(\alpha, \beta).$$

On this basis, one can assert that

$$U(\alpha, \beta) = 1 + \frac{1}{2\pi i}\cdot\frac{\gamma}{\Pi\left(\frac{\gamma}{2}\right)}\int_{-i\infty}^{+i\infty}\frac{\Pi(\omega)\cos\sqrt{\omega^2-\frac{\gamma^2}{4}}\,\alpha}{\sqrt{\omega^2-\frac{\gamma^2}{4}}\sin\sqrt{\omega^2-\frac{\gamma^2}{4}}}\,e^{\left(\omega-\frac{\gamma}{2}\right)\beta}\,d\omega \tag{20}$$

and the problem is completely solved.

It is of interest to investigate the concentration distribution for small positive β when $\alpha = \pm 1$. Let $\Pi(-\omega)$ be a function constructed analogously to the function $\Pi(\omega)$ with respect to those negative roots of the functions $\sin\sqrt{\omega^2-\gamma^2/4}$ and $\cos\sqrt{\omega^2-\gamma^2/4}$ which lie to the left of the line $\omega = -\gamma/2$. As is easily seen, the functions $\Pi(\omega)$ and $\Pi(-\omega)$ satisfy the relation

$$\Pi(\omega)\,\Pi(-\omega) = \frac{\sin\sqrt{\omega^2-\frac{\gamma^2}{4}}}{\sqrt{\omega^2-\frac{\gamma^2}{4}}\cos\sqrt{\omega^2-\frac{\gamma^2}{4}}}.$$

Hence

$$\frac{\Pi(\omega)\sqrt{\omega^2-\frac{\gamma^2}{4}}\cos\sqrt{\omega^2-\frac{\gamma^2}{4}}}{\sin\sqrt{\omega^2-\frac{\gamma^2}{4}}} = \frac{1}{\Pi(-\omega)}.$$

Therefore,

$$U(1, \beta) = 1 + \frac{1}{2\pi i}\cdot\frac{\gamma}{\Pi\left(\frac{\gamma}{2}\right)}\int_{-i\infty}^{+i\infty}\frac{1}{\omega^2-\frac{\gamma^2}{4}}\cdot\frac{e^{\left(\omega-\frac{\gamma}{2}\right)\beta}}{\Pi(-\omega)}\,d\omega.$$

Alternatively, considering that we are looking at positive β,

$$U(1, \beta) = 1 + \frac{1}{2\pi i}\oint_{\Gamma^+}\frac{\gamma}{\Pi\left(\frac{\gamma}{2}\right)}\cdot\frac{1}{\Pi(-\omega)\left(\omega^2-\frac{\gamma^2}{4}\right)}\,e^{\left(\omega-\frac{\gamma}{2}\right)\beta}\,d\omega$$

$$+\frac{1}{2\pi i}\int_{-\frac{\gamma}{2}-i\infty}^{\left(-\frac{\gamma}{2}-\right)-\frac{\gamma}{2}+i\infty}\frac{\gamma}{\Pi\left(\frac{\gamma}{2}\right)}\cdot\frac{1}{\Pi(-\omega)\left(\omega^2-\frac{\gamma^2}{4}\right)}\,e^{\left(\omega-\frac{\gamma}{2}\right)\beta}\,d\omega.$$

Here, Γ^+ is a closed contour which surrounds the point $\omega = -\gamma/2$, which is traversed in the positive direction, and is so small that in the finite region of the ω plane bounded by it there are no poles of the function inside the integral sign other than $\omega = -\gamma/2$, and the symbol $(-\gamma/2-)$ indicates bypassing the point $\omega = -\gamma/2$ on the left.

One can show that the integral along the contour Γ^+ equals $-(e^{-\gamma\beta}/\Pi^2(\gamma/2))$. Therefore,

$$U(1, \beta) = 1 - \frac{e^{-\gamma\beta}}{\Pi^2\left(\frac{\gamma}{2}\right)} + \frac{1}{2\pi i}\int_{-i\infty}^{(0-)+i\infty} \frac{\gamma}{\Pi\left(\frac{\gamma}{2}\right)} \frac{e^{-\gamma\beta} e^{u\beta}\, du}{u(u-\gamma)\Pi\left[-\left(u-\frac{\gamma}{2}\right)\right]}$$

$$= 1 - \frac{e^{-\gamma\beta}}{\Pi^2\left(\frac{\gamma}{2}\right)} + \frac{e^{-\gamma\beta}}{2\pi i}\int_{-i\infty}^{(0-)+i\infty} \frac{\gamma}{\Pi\left(\frac{\gamma}{2}\right)} \cdot \frac{e^{-u\beta}\, du}{u(u+\gamma)\Pi\left(u+\frac{\gamma}{2}\right)}.$$

Setting $u\beta = v$ $(\beta > 0)$ in the last integral, we have

$$U(1, \beta) = 1 - \frac{e^{-\gamma\beta}}{\Pi^2\left(\frac{\gamma}{2}\right)} - \frac{e^{-\gamma\beta}}{2\pi i}\int_C \frac{\gamma}{\Pi\left(\frac{\gamma}{2}\right)} \frac{e^{-v} dv}{v\left(\frac{v}{\beta}+\gamma\right)\Pi\left(\frac{v}{\beta}+\frac{\gamma}{2}\right)}.$$

In the last equation, C is a contour consisting of the axis $\operatorname{Re} v = 0$ with the exception of the segment $(-ir, +ir)$, where r is a fixed positive number, and a semicircle of radius r with its center at the point $v = 0$ going around the origin on the left. Considering that $|v| \geq r$ on the contour C and the asymptotic relation (16), we can write for β close to zero

$$U(1, \beta) \sim 1 - \frac{e^{-\gamma\beta}}{\Pi^2\left(\frac{\gamma}{2}\right)} - \frac{\gamma\sqrt{\beta}\, e^{-\gamma\beta}}{2\pi\Pi\left(\frac{\gamma}{2}\right)}\int_C \frac{e^{-v}\, dv}{v\sqrt{v}}.$$

If L is a path which goes around the origin without intersecting the contour C and passes along the positive semiaxis $\operatorname{Im} v = 0$, then

$$\int_C \frac{e^{-v}\, dv}{v\sqrt{v}} = -\int_L \frac{e^{-v}\, dv}{v\sqrt{v}} = 2\Gamma\left(-\frac{1}{2}\right) = -4\sqrt{\pi}.$$

Consequently,

$$U(1, \beta) \sim 1 - \left(\frac{1}{\Pi\left(\frac{\gamma}{2}\right)} - \frac{2\gamma\sqrt{\beta}}{\sqrt{\pi}}\right)\frac{e^{-\gamma\beta}}{\Pi\left(\frac{\gamma}{2}\right)}.$$

NOTATION

$C(\alpha, \beta)$	is the concentration of diffusing substance;
C_0	is the concentration of reagent at the entrance to the cathode channel of the transformer;
D	is the diffusivity;
$\bar{v}$	is the convection rate;
v	is the modulus of the convection rate;
h	is the width of a band, a constant having the dimensions of length;
$\gamma = vh/D$	is a positive dimensionless constant.

LITERATURE CITED

1. A. M. Danilevskii, "Current distribution in a cylindrical electrode," Trans. Scientific Research Institute for Mathematics and Mechanics, Kharkov State University [in Ukrainian], Vol. 13, No. 1 (1936).
2. M. L. Fish, Chemotron Instruments in Automation [in Russian], Tekhnika, Kiev (1967).